Day 22

Bayes and Kalman Filter

Combining Two Noisy Measurements

- recall from the last lecture that the minimum variance estimate for combining two noisy measurements

$$\mu = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} x_1 + \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} x_2 = \underbrace{x_1}_{\text{previous measurement}} + \underbrace{\frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}}_{\text{Kalman gain}} \underbrace{(x_2 - x_1)}_{\text{measurement difference}}$$

$$\operatorname{var}(\mu) = \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

- claim: the estimate is a special case of the discrete Kalman filter algorithm

Discrete Kalman Filter

- estimates the state x of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

plant model
process model

with a measurement

$$z_t = C_t x_t + \delta_t$$

measurement model
observation model

Components of a Kalman Filter

A_t Matrix (nxn) that describes how the state evolves from t to $t\text{-}1$ without controls or noise.

B_t Matrix (nxl) that describes how the control u_t changes the state from t to $t\text{-}1$.

C_t Matrix (kxn) that describes how to map the state x_t to an observation z_t.

ε_t

δ_t

Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance R_t and Q_t respectively.

Kalman Filter Algorithm

1. Algorithm **Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

2. Prediction:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

Combining Two Noisy Measurements

- combining two noisy measurements of a fixed scalar quantity is a static 1D-state estimation problem
  - the state does not evolve as a function of time and does not depend on any control input

$$A_t = 1, \quad B_t = 0, \quad R_t = 0 \qquad \begin{aligned} x_t &= A_t x_{t-1} + B_t u_t + \varepsilon_t \\ &= x_{t-1} \end{aligned}$$

- our measurements are direct (noisy) measurements of the state

$$C_t = 1, \quad Q_t = \sigma_t^2 \qquad z_t = x_t + \delta_t$$

Combining Two Noisy Measurements

- start by initializing the Kalman filter with the first measurement and its variance

estimated state $$\mu_1 = x_1$$

estimated state covariance $$\Sigma_1 = \sigma_1^2$$

- now substitute into the Kalman filter algorithm

Plant or Process Model

- describes how the system state changes as a function of time, control input, and noise

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

  - x_t state at time t
  - u_t control inputs at time t
  - ε_k process noise at time t (assumed Gaussian with covariance R_t)
  - A_t state transition model or matrix at time t
  - B_t control-input model or matrix at time t
- note that the model is linear and assumes additive Gaussian noise

Example: Omnidirectional Robot

- an omnidirectional robot is a robot that can move in any direction (constrained in the ground plane)
  - http://www.youtube.com/watch?v=DPz-ullMOqc
  - http://www.engadget.com/2011/07/09/curtis-boirums-robotic-car-makes-omnidirectional-dreams-come-tr/

- if we are not interested in the orientation of the robot then its state is simply its location

$$x_t = \begin{bmatrix} x \\ y \end{bmatrix}_t$$

Example: Omnidirectional Robot

- a possible choice of motion control is simply a change in the location of the robot

$$x_t = \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}_{t-1}}_{x_{t-1}} + \underbrace{\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}_t}_{u_t}$$

- with noisy control inputs

$$x_t = \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}_{t-1}}_{x_{t-1}} + \underbrace{\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}_t}_{u_t} + \varepsilon_t$$

Measurement Model

- describes how sensor measurements vary as a function of the system state

$$z_t = C_t x_t + \delta_t$$

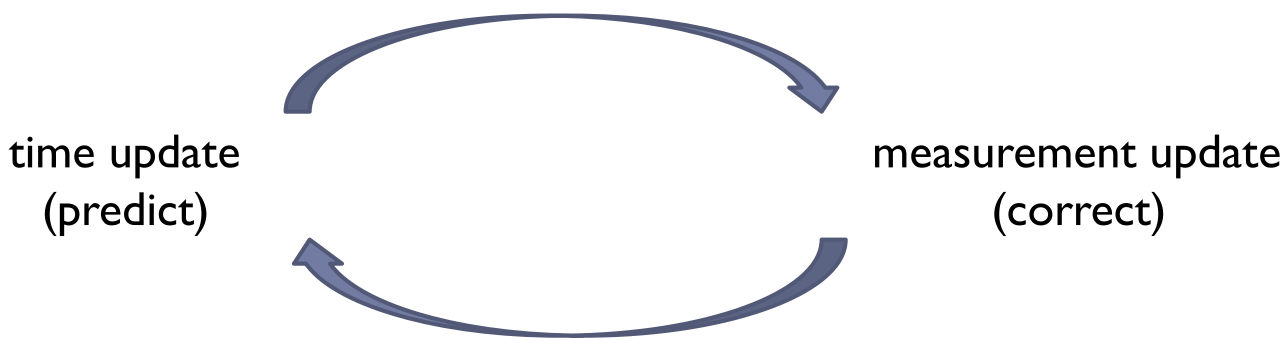
  - z_t sensor measurement at time t
  - δ_t sensor noise at time t (assumed Gaussian with covariance Q_t)
  - C_t observation model or matrix
- notice that the model is linear and assumes additive Gaussian noise

Kalman Filter

- the Kalman filter is a provably optimal (in terms of least-squared error) algorithm for fusing sensor measurements to produce an estimate of the state and the state covariance
  - x_t state at time t
  - Σ_t state covariance at time t

Kalman Filter

- the Kalman filter estimates a process in two stages
  1. **prediction:** current state and state covariance estimates are projected forward in time to predict the new state and state covariance
     - "time update equations"
  2. **correction:** the sensor measurements are incorporated into the predicted state to obtain improved estimates of the state and state covariance
     - "measurement update equations"



Kalman Filter Algorithm

1. **Initialization**
   - choose (guess) initial values for mean state and state covariance estimates

$$\mu_0$$

$$\Sigma_0$$

Kalman Filter Algorithm

2. **Prediction:**
   - predict the next state using the plant model

$$\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$$

   - predicted state covariance grows (because we are not incorporating the sensor measurements yet)

$$\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

     - R_t covariance of the plant noise

Kalman Filter Algorithm

3. **Correction:** correct the predicted state using the sensor measurement
   - expected value of measurements (from measurement model)

     $$\bar{z}_t = C_t \bar{\mu}_t$$

   - difference between actual and expected measurements

     $$r_t = z_t - \bar{z}_t$$

   - measurement covariance

     $$S_t = C_t \, \bar{\Sigma}_t \, C_t^T + Q_t$$

   - Kalman gain

     $$K_t = \bar{\Sigma}_t \, C_t^T \, S_t^{-1}$$

Kalman Filter Algorithm

4. **State and state covariance:**

- new state estimate incorporating most recent measurement

$$\mu_t = \bar{\mu}_t + K_t \, r_t$$

- new state covariance estimate

$$\Sigma_t = \left(I - K_t \, C_t \right) \bar{\Sigma}_t$$